15.7 Lecture 3: Function bounds for ρ

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Last class

In spherical coordinates, ρ keeps track of the distance from the origin, ϕ keeps track of the angle from the positive z-axis, and θ keeps track of the angle from the positive z-axis to the projection of the point in the xy-plane.

We have the following relationships:
$x^2 + y^2 + z^2 = \rho^2$
$z = \rho \cos(\phi)$
$y = \rho \sin(\phi) \sin(\theta)$
$x = \rho \sin(\phi) \cos(\theta)$

Strategy for finding integration bounds

1. Sketch the region
2. Find ρ-limits using rays from the origin.
3. Find ϕ-limits.
4. Find θ-limits.

It is often helpful to do steps 1-3 in a 2D plane where θ is fixed. If the functions involved depend on θ, then this might be a bad choice, but usually the functions will depend on ρ and ϕ, meaning θ is free and the picture looks the same no matter what θ is.

Problem #35, section 15.7

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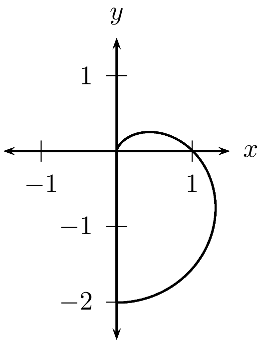

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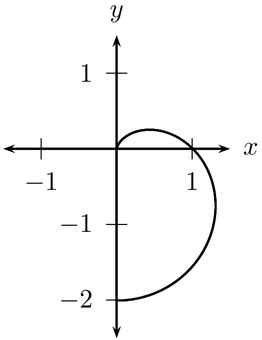
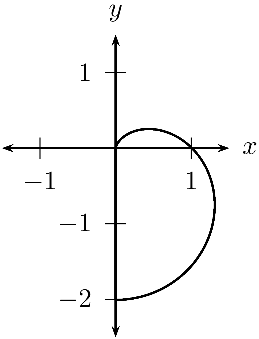


We see that since the origin is in the region, the lower bound on ρ is 0. Any ray from the origin that enters the region leaves the region on the cardioid, which is given by $\rho = 1 - \cos(\phi)$. So this is an upper bound for ρ.

$$0 \leq \rho \leq 1 - \cos(\phi)$$

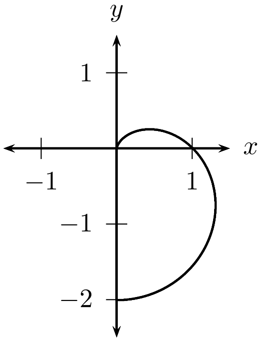
ϕ and θ bounds



Since the cardioid is graphed from the positive z-axis to the negative z-axis, the bounds on ϕ are 0 and π.

Since the problem states no restriction on any of the variables, we have $0 \leq \theta \leq 2\pi$.

ϕ and θ bounds



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Thus the volume is given by the integral below:

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=1-\cos(\phi)} \rho^2 \sin(\phi) \; d\rho \; d\phi \; d\theta$$

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This problem should really be done in cylindrical coordinates, since the bounds are readily obtained (though there is some work to do to find the shadow):

$$r^2 \leq z \leq \sqrt{2 - r^2}$$

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

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However, we will do this problem in spherical coordinates as an example of setting up 2 regions of integration.

Example 2 cont.

Example

Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ *and below by the paraboloid* $z = x^2 + y^2$*.*

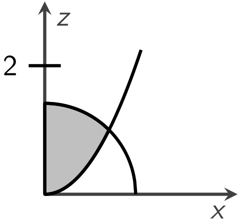
Before we get to bounds, we will simplify the equations in question. The sphere is just the equation $\rho^2 = 2$, or $\rho = \sqrt{2}$. We have seen in a previous class that $x^2 + y^2 = \rho^2 \sin^2(\phi)$. Thus the paraboloid is $z = \rho^2 \sin^2(\phi)$, or $\rho \cos(\phi) = \rho^2 \sin^2(\phi)$. Dividing by ρ and solving for ρ, we find the paraboloid is given by $\rho = \frac{\cos(\phi)}{\sin^2(\phi)}$.

Sketch

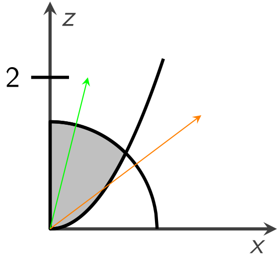
Example

Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

Since both the sphere and the paraboloid are symmetric about the z-axis, it doesn't matter what θ value we choose to look at, so we will look at $\theta = 0$ over the positive x-axis. Thus the sphere is $x^2 + z^2 = 2$ and the paraboloid is $z = x^2$.



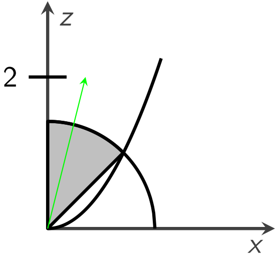
ρ-bounds

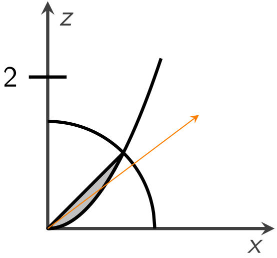


Some rays from the origin leave the region on the sphere, like the green ray shown. Some leave the region on the paraboloid, like the orange line shown. Thus we will have to break the region up into 2 pieces to find the ρ-bounds.

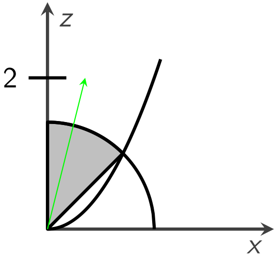
The type of ray changes from green to orange at the intersection of the sphere and paraboloid. To find this point, we set the two equations equal to each other: $x^2 = \sqrt{2 - x^2}$ or $x^4 = 2 - x^2$, which has solutions $x = \pm 1$. Thus the point is $(x, z) = (1, \sqrt{2 - 1^2}) = (1, 1)$.

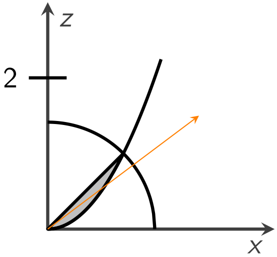
ρ-bounds cont.





ρ-bounds cont.





The green ray leaves the region when $\rho = \sqrt{2}$, so we have the bounds

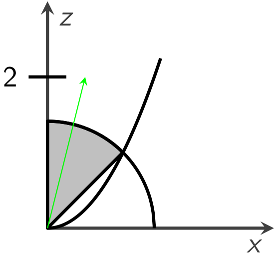
$$0 \leq \rho \leq \sqrt{2}$$

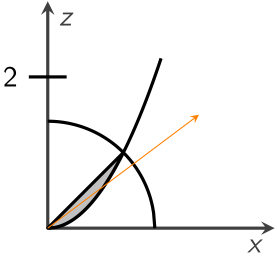
The orange ray leaves the region when $\rho = \frac{\cos(\phi)}{\sin^2(\phi)}$, so we have the bounds

$$0 \leq \rho \leq \frac{\cos(\phi)}{\sin^2(\phi)}$$

ϕ and θ bounds

Since $\phi = \pi/4$ at $(1, 0, 1)$, we have the following bounds on ϕ.





$$0 \leq \rho \leq \sqrt{2}$$

$$0 \leq \phi \leq \pi/4$$

$$0 \leq \rho \leq \frac{\cos(\phi)}{\sin^2(\phi)}$$

$$\pi/4 \leq \phi \leq \pi/2$$

Since the problem had no constraints on θ, we can take the bounds $0 \leq \theta \leq 2\pi$ for both pieces of the region.

Example 2 conclusion

Example

Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

Based on the bounds we found, we have that the volume is

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/4} \int_{\rho=0}^{\rho=\sqrt{2}} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$+ \int_{\theta=0}^{\theta=2\pi} \int_{\phi=\pi/4}^{\phi=\pi/2} \int_{\rho=0}^{\rho=\frac{\cos(\phi)}{\sin^2(\phi)}} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$